

COMPOSITION FACTORS OF KAC-MODULES FOR THE GENERAL LINEAR LIE SUPERALGEBRAS $\mathfrak{gl}_{m|n}$

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ABSTRACT. The composition factors of Kac-modules for the general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ are explicitly determined. In particular, a conjecture of Hughes, King and van der Jeugt in [*J. Math. Phys.*, **41** (2000), 5064-5087] is proved.

1. INTRODUCTION

Following the classification of simple Lie superalgebras [7, 8], Kac studied finite-dimensional modules of the classical Lie superalgebras [9, 10], distinguishing between typical and atypical modules. He also introduced what is now called the Kac-module V_λ , which was shown to be simple if and only if λ is typical. Since then, Kac-modules, which themselves encapsulate rich information on the structure of the representations, have been playing extremely active roles in the representation theory of Lie superalgebras. For λ atypical, the structure of V_λ , or more generally the problem of classifying finite-dimensional indecomposable modules has been the subject of intensive study (see, e.g., the References). By analyzing structures of Kac-modules, van der Jeugt [22] constructed a character formula for all finite-dimensional irreducible modules over the orthosymplectic Lie superalgebras $\mathfrak{osp}_{2|2n}$. However in the case of the general linear Lie superalgebras $\mathfrak{gl}_{m|n}$, it turned out that the analysis of structures of Kac-modules is a technical and difficult problem.

There were many partial results on describing structures of Kac-modules or determining character formulae of irreducible modules over $\mathfrak{gl}_{m|n}$. However the full problem remained open until Serganova [16, 17], based on ideas from Kazhdan-Lusztig theory, derived an algorithm for computing character formulae for irreducible modules L_λ , and determining the multiplicities $a_{\lambda,\mu} = [V_\lambda : L_\mu]$ of composition factors L_μ of Kac-modules V_λ . (The implementation of this algorithm turned out to be rather unwieldy to use, e.g., a fact which was conjectured by van der Jeugt and Zhang [25] and proved by Brundan [2] that the composition multiplicities $a_{\lambda,\mu}$ of the Kac-modules are all either 0 or 1, does not seem to follow easily from Serganova's formula since that involves certain alternating sums.) This work was further developed in [2], where Brundan used quantum group techniques to develop a very practicable algorithm for computing Kazhdan-Lusztig polynomials for finite-dimensional irreducible modules over $\mathfrak{gl}_{m|n}$ and proved a theorem previously conjectured in [25], which determines all weights λ such that $a_{\lambda,\mu} = 1$, for a given μ (there are precisely 2^r such λ 's, where r is the degree of atypicality of μ). This algorithm was further implemented by Zhang and the author [21], who obtained some closed formulae to compute Kazhdan-Lusztig polynomials, characters and dimensions for all finite-dimensional irreducible modules over $\mathfrak{gl}_{m|n}$.

Brundan's result is useful in understanding structures of Kac-modules. However this result is not ready to be used in describing the structure of a given Kac-module as clear

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as one would wish, it still seems to be a problem on how to explicitly determine the composition factors L_μ of the Kac-module V_λ , for a given λ . Due to the crucial role that Kac-modules have been playing in the representation theory of Lie superalgebras, it seems to us that it is highly desirable to derive a closed formula for computing the composition factors of V_λ . Hughes, King and van der Jeugt [6] described an algorithm to determine all the composition factors of Kac-modules for $\mathfrak{gl}_{m|n}$. They conjectured that there exists a bijection between the composition factors of V_λ and certain permissible codes (see Definition 4.2). This conjecture, which withstood extensive tests against computer calculations for a wide range of weights λ , describes clearly the structure of V_λ .

In this paper, we shall further implement Brundan's result to determine explicitly the composition factors of the finite-dimensional Kac-modules over the general linear Lie superalgebras. A closed formula is obtained for determining the set of the composition factors of V_λ (see Theorem 3.13). This result is quite explicit and easy to apply. In particular we are able to prove the conjecture of Hughes et al (see Theorem 4.9). The techniques used in the paper are purely combinatorial.

The organization of the paper is as follows. Some background material on $\mathfrak{gl}_{m|n}$ which will be used in the paper is recalled in Section 2. In Section 3, the notion of **nqc**-relationship is introduced that is crucial in the proof of the main theorem, which is also presented in this section. Section 4 is devoted to a proof of the conjecture of Hughes et al after the notion of permissible codes being introduced, and the final section is devoted to the proof of the main result. Finally we may like to mention that, as is stated earlier, due to the fact that a Kac-module itself has a complicated structure, some arguments in the proof may render technical.

2. PRELIMINARIES

Denote by $\mathbb{C}^{m|n}$ the \mathbb{Z}_2 -graded vector space with even subspace \mathbb{C}^m and odd subspace \mathbb{C}^n . Then $\text{End}_{\mathbb{C}}(\mathbb{C}^{m|n})$ with the \mathbb{Z}_2 -graded commutator forms the *general linear superalgebra* $\mathfrak{gl}_{m|n}$, which is denoted by \mathfrak{g} throughout the paper. Choose a basis $\{v_a \mid a \in \mathbf{I}\}$, for $\mathbb{C}^{m|n}$, where $\mathbf{I} = \{1, 2, \dots, m+n\}$, and v_a is even if $a \leq m$, and odd otherwise. Let E_{ab} be the matrix unit, namely, the $(m+n) \times (m+n)$ -matrix with all entries being zero except that at the (a, b) position which is 1. Then $\{E_{ab} \mid a, b \in \mathbf{I}\}$ forms a basis of \mathfrak{g} , with E_{ab} being even if $a, b \leq m$, or $a, b > m$, and odd otherwise. Define the map

$$[\] : \mathbf{I} \rightarrow \mathbb{Z}_2, \quad [a] = \begin{cases} \bar{0}, & \text{if } a \leq m, \\ \bar{1}, & \text{if } a > m. \end{cases}$$

Then the commutation relations can be written as

$$[E_{ab}, E_{cd}] = E_{ad}\delta_{bc} - (-1)^{([a]-[b])([c]-[d])}E_{cb}\delta_{ad}.$$

The upper triangular matrices form a *Borel subalgebra* \mathfrak{b} of \mathfrak{g} , which contains the *Cartan subalgebra* \mathfrak{h} of diagonal matrices. Let $\{\epsilon_a \mid a \in \mathbf{I}\}$ be the basis of \mathfrak{h}^* such that $\epsilon_a(E_{bb}) = \delta_{ab}$. The supertrace induces a *bilinear form* $(\ , \) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ on \mathfrak{h}^* such that $(\epsilon_a, \epsilon_b) = (-1)^{[a]}\delta_{ab}$. Relative to the Borel subalgebra \mathfrak{b} , the roots of \mathfrak{g} can be expressed as $\epsilon_a - \epsilon_b$, $a \neq b$, where $\epsilon_a - \epsilon_b$ is even if $[a] + [b] = \bar{0}$ and odd otherwise. The set of the *positive roots* is $\Delta^+ = \{\epsilon_a - \epsilon_b \mid a < b\}$, and the set of *simple roots* is $\{\epsilon_a - \epsilon_{a+1} \mid a < m+n\}$.

We denote $\mathbf{I}^1 = \{1, 2, \dots, m\}$ and $\mathbf{I}^2 = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$, where here and below we use the notation

$$\bar{\nu} = \nu + m.$$

Then $\mathbf{I} = \mathbf{I}^1 \cup \mathbf{I}^2$. The sets of *positive even roots* and *odd roots* are respectively

$$\begin{aligned}\Delta_0^+ &= \{\alpha_{i,j} = \epsilon_i - \epsilon_j, \alpha_{\nu,\eta} = \epsilon_\nu - \epsilon_\eta \mid 1 \leq i < j \leq m, \bar{1} \leq \nu < \eta \leq \bar{n}\}, \\ \Delta_1^+ &= \{\alpha_{i,\nu} = \epsilon_i - \epsilon_\nu \mid i \in \mathbf{I}^1, \nu \in \mathbf{I}^2\}.\end{aligned}$$

The Lie algebra \mathfrak{g} admits a \mathbb{Z}_2 -consistent \mathbb{Z} -grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \text{ where } \mathfrak{g}_0 = \mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \text{ and } \mathfrak{g}_{\pm 1} \subset \mathfrak{g}_{\bar{1}},$$

with \mathfrak{g}_{+1} (resp. \mathfrak{g}_{-1}) being the nilpotent subalgebra spanned by the odd positive (resp. negative) root spaces. We define a total order on Δ_1^+ by

$$\alpha_{i,\nu} < \alpha_{j,\eta} \iff \nu - i < \eta - j \text{ or } \nu - i = \eta - j \text{ but } i > j.$$

An element in \mathfrak{h}^* is called a *weight*. A weight Λ is *integral* if $(\Lambda, \epsilon_a) \in \mathbb{Z}$ for all a , and *dominant* if $2(\Lambda, \alpha)/(\alpha, \alpha) \geq 0$ for all positive even roots α of \mathfrak{g} . Denote by P (resp. P_+) the set of integral (resp. dominant integral) weights.

Since $\{\epsilon_i \mid i \in \mathbf{I}\}$ is a \mathbb{C} -basis of \mathfrak{h}^* , a weight $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = \sum_{i \in \mathbf{I}} \lambda'_i \epsilon_i$ with $\lambda'_i \in \mathbb{C}$, and it is usually denoted by

$$\lambda = (\lambda'_1, \lambda'_2, \dots, \lambda'_m \mid \lambda'_1, \lambda'_2, \dots, \lambda'_{\bar{n}}). \quad (2.1)$$

But sometimes, we shall find it is more convenient to denote the weight λ by

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m \parallel \lambda_{\bar{1}}, \lambda_{\bar{2}}, \dots, \lambda_{\bar{n}}), \text{ where } \lambda_i = \begin{cases} \lambda'_i + i & \text{if } i \in \mathbf{I}_1, \\ -\lambda'_i + i - m & \text{if } i \in \mathbf{I}_2. \end{cases} \quad (2.2)$$

One can easily convert notation (2.1) to notation (2.2), or vice versa. With notation (2.2), the set of integral weights coincides with the set $\mathbb{Z}^{m|n}$ of $(m+n)$ -tuples of integers and the set of dominant integral weights coincides with the subset $\mathbb{Z}_+^{m|n}$ of $(m+n)$ -tuples λ satisfying

$$\lambda_1 > \lambda_2 > \dots > \lambda_m, \quad \lambda_{\bar{1}} < \lambda_{\bar{2}} < \dots < \lambda_{\bar{n}}. \quad (2.3)$$

We should note that there is no loss of generality in restricting our attention to integral weights λ since an arbitrary “ r -fold atypical” (see below) finite-dimensional Kac-module can be obtained from V_λ , for some $\lambda \in \mathbb{Z}^{m|n}$, by tensoring with one-dimensional module.

Let $W = S_m \times S_n$ be the *Weyl group* of \mathfrak{g} , where S_m is the *symmetric group* of degree m . The action of W on P , by definition, is

$$w\lambda = (\lambda_{w(1)}, \lambda_{w(2)}, \dots, \lambda_{w(m)} \parallel \lambda_{w(\bar{1})}, \lambda_{w(\bar{2})}, \dots, \lambda_{w(\bar{n})}) \in \mathbb{Z}^{m|n} \text{ for } \lambda \in \mathbb{Z}^{m|n}, \quad (2.4)$$

where $w \in W$. An integral weight λ is called *regular* or *non-vanishing* (in sense of [6, 24]) if it is W -conjugate to a dominant weight (which is denoted by λ^+ throughout the paper), otherwise it is called *vanishing*.

For a regular weight λ in (2.2), we define the *atypicality matrix* of λ to be the $m \times n$ matrix

$$A(\lambda) = (A(\lambda)_{i,\eta})_{m \times n}, \text{ where } A(\lambda)_{i,\eta} = \lambda_i - \lambda_{\bar{\eta}}, \quad 1 \leq i \leq m, \quad 1 \leq \eta \leq n. \quad (2.5)$$

An odd root $\alpha_{i,\bar{\eta}}$ is an *atypical root* of λ if $A(\lambda)_{i,\eta} = 0$. Let $\Gamma_\lambda = \{\alpha_{i,\bar{\eta}} \mid A(\lambda)_{i,\eta} = 0\}$ be the set of atypical roots, and $r = \#\Gamma_\lambda$ be the *degree of atypicality*. We also denote $\#\lambda = r$. Thus $\#\lambda$ is the number of pairs $(i, \bar{\eta})$ whose entries are equal: $\lambda_i = \lambda_{\bar{\eta}}$. A weight λ is *typical* if $r = 0$; *atypical* if $r > 0$ (in this case λ is also called an r -fold

atypical weight). If λ is dominant and r -fold atypical, we label its atypical roots by $\gamma_1, \dots, \gamma_r$ ordered in such a way that (cf. Example 3.3)

$$\begin{cases} \gamma_1 < \gamma_2 < \dots < \gamma_r \text{ and } \gamma_s = \alpha_{m_s, n_s}, & s = 1, 2, \dots, r, \text{ with} \\ 1 \leq m_r < m_{r-1} < \dots < m_1 \leq m < n_1 < n_2 < \dots < n_r \leq \bar{n}. \end{cases} \quad (2.6)$$

For an integral dominant weight λ , denote by $L_\lambda^{(0)}$ the finite-dimensional irreducible \mathfrak{g}_0 -module with highest weight λ . Extend it to a $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ -module by putting $\mathfrak{g}_{+1} L_\lambda^{(0)} = 0$. Then the *Kac-module* V_λ is the induced module

$$V_\lambda = \text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}^{\mathfrak{g}} L_\lambda^{(0)} \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} L_\lambda^{(0)}.$$

Denote by L_λ the *irreducible module* with highest weight λ (which is the unique irreducible quotient module of V_λ).

The following result is due to Kac [9, 10].

Theorem 2.1. *The finite-dimensional Kac-module V_λ over \mathfrak{g} is irreducible if and only if λ is typical.*

3. COMPOSITION FACTORS OF KAC-MODULES

For convenience, we introduce the notation

$$[s, t] = \begin{cases} \{i \in \mathbb{Z} \mid s \leq i \leq t\} & \text{if } s \leq t, \\ \emptyset & \text{otherwise,} \end{cases}$$

for $s, t \in \mathbb{Z}$ (this notation will not be confused with the Lie bracket since we do not need to use that below).

3.1. The nqc -relationship. Let P_r be the subset of $\mathbb{Z}^{m|n}$ consisting of regular r -fold atypical weights λ such that the atypical roots of λ satisfy (2.6) and

$$\lambda_i > \lambda_j, \quad i < j \leq m, \quad i, j \in \mathbf{I} \setminus \mathbf{I}_r^\lambda \quad \text{and} \quad \lambda_a < \lambda_b, \quad m < a < b, \quad a, b \in \mathbf{I} \setminus \mathbf{I}_r^\lambda, \quad (3.1)$$

where $\mathbf{I}_r^\lambda = \{m_r, m_{r-1}, \dots, m_1, n_1, n_2, \dots, n_r\}$. We denote by D_r the subset of P_r of the elements λ such that $\lambda_{m_1} < \lambda_{m_2} < \dots < \lambda_{m_r}$.

For $\lambda \in P_r$, we denote

$$t^\lambda \in \mathbb{Z}^{r|r} \quad \text{or} \quad \bar{t}^\lambda \in \mathbb{Z}^{m-r|n-r} \quad (3.2)$$

to be the element obtained from λ by deleting its i -th entry for $i \in \mathbf{I} \setminus \mathbf{I}_r^\lambda$ or $i \in \mathbf{I}_r^\lambda$ respectively. Thus \bar{t}^λ is always dominant for all $\lambda \in P_r$, and t^λ is dominant if and only if $\lambda \in D_r$. We also introduce the following three sets of integers:

$$S(\lambda) = \{\lambda_i \mid i \in \mathbf{I}\}, \quad T(\lambda) = S(t^\lambda) = \{\lambda_{m_s} \mid s \in [1, r]\}, \quad \bar{T}(\lambda) = S(\bar{t}^\lambda) = S(\lambda) \setminus T(\lambda). \quad (3.3)$$

Convention 3.1. *We usually use the superscript ' λ ' to indicate that a notation is associated with λ , like in \mathbf{I}_r^λ , t^λ , \bar{t}^λ . However, when confusion is unlikely to occur, the superscript will be dropped, like in $\ell_{s,t}$, $c_{s,t}$, k_i , \check{k}_i below.*

The following notion of nqc -relationship was first introduced by Hughes et al [6] from a different point of view.

Definition 3.2. *Suppose $\lambda \in D_r$. Let $\mathbf{n}, \mathbf{q}, \mathbf{c}$ be three symbols. For $1 \leq s \leq t \leq r$, we define*

$$\ell_{s,t} = \#([f_{m_s}, f_{m_t}] \setminus S(f)), \quad c_{s,t} = \begin{cases} \mathbf{n} & \text{if } \ell_{s,t} > t - s, \\ \mathbf{q} & \text{if } \ell_{s,t} = t - s, \\ \mathbf{c} & \text{if } \ell_{s,t} < t - s. \end{cases} \quad (3.4)$$

Note that in (3.4), we abused the notation by using $[f_{m_s}, f_{m_t}] \setminus S(f)$ to denote

$$[f_{m_s}, f_{m_t}] \setminus ([f_{m_s}, f_{m_t}] \cap S(f)).$$

Two atypical roots γ_s, γ_t of λ are called (cf. [6, 24])

- (i) normally related or **n**-related $\iff c_{s,t} = \mathbf{n}$;
- (ii) quasi-critically related or **q**-related $\iff c_{s,t} = \mathbf{q}$;
- (iii) critically related or **c**-related $\iff c_{s,t} = \mathbf{c}$.

Note that **q**-relationship is reflexive and transitive but not symmetric ($c_{t,s}$ is not defined when $t > s$); **c**-relationship or **n**-relationship is transitive.

Example 3.3. Suppose

$$\lambda = \overset{4}{(15, 11, 10, 7, 6, 4, 3} \parallel \overset{1}{3, 5, 7, 8, 10, 15}), \quad S(\lambda) = \overset{1}{\{3, 4, 5, 6, 7, 8, 10, 11, 15\}}, \quad (3.5)$$

where we put a label s over an entry to indicate it corresponds to the s -th atypical root (such an entry is called an atypical entry). Then

- $c_{1,2} = \mathbf{c}$ since $[3, 7] \setminus S(\lambda) = \emptyset$ is of cardinality $0 < 1$;
- $c_{1,3} = \mathbf{c}$ since $[3, 10] \setminus S(\lambda) = \{9\}$ is of cardinality $1 < 2$;
- $c_{1,4} = \mathbf{n}$ since $[3, 15] \setminus S(\lambda) = \{9, 12, 13, 14\}$ is of cardinality $4 > 3$; and
- $c_{2,3} = \mathbf{q}$, $c_{2,4} = \mathbf{n}$, $c_{3,4} = \mathbf{n}$.

A simple way to determine $c_{s,t}$ is to count the number of integers between λ_{m_s} and λ_{m_t} which do not belong to the set $S(\lambda)$. If the number (which is $\ell_{s,t}$) is smaller than (resp. equal to, or bigger than) $t - s$ then $c_{s,t} = \mathbf{c}$ (resp. **q**, or **n**).

The following discussion may illustrate the significance of the concept of **nqc**-relationship.

3.2. Raising and lowering operators. Let $\lambda \in D_r$. For $s = 1, 2, \dots, r$, we set (recall Convention 3.1)

$$p_s = \begin{cases} s & \text{if } s = r \text{ or } c_{s,s+1} \neq \mathbf{c}, \\ \max\{p \in [s+1, r] \mid c_{s,s+1} = c_{s,s+2} = \dots = c_{s,p} = \mathbf{c}\} & \text{otherwise,} \end{cases} \quad (3.6)$$

$$\check{p}_s = \begin{cases} s & \text{if } s = 1 \text{ or } c_{s-1,s} \neq \mathbf{c}, \\ \min\{p \in [1, s-1] \mid c_{p,s} = c_{p+1,s} = \dots = c_{s-1,s} = \mathbf{c}\} & \text{otherwise.} \end{cases} \quad (3.7)$$

Namely, p_s (resp. \check{p}_s) is the largest (resp. smallest) integer such that all γ_i with $i \in [s+1, p_s]$ (resp. $i \in [\check{p}_s, s-1]$) are **c**-related to γ_s . Define r -tuples (k_1, k_2, \dots, k_r) and $(\check{k}_1^{(\nu)}, \check{k}_2^{(\nu)}, \dots, \check{k}_r^{(\nu)})$, $\nu \geq 1$, of positive integers associated with $\lambda \in D_r$ by:

$$k_s = \min \left\{ k > 0 \mid \# \left([\lambda_{m_s}, \lambda_{m_s} + k] \setminus S(\lambda) \right) = p_s + 1 - s \right\}, \quad (3.8)$$

$$\check{k}_s^{(\nu)} = \min \left\{ k > 0 \mid \# \left([\lambda_{m_s} - k, \lambda_{m_s}] \setminus S(\lambda) \right) = \nu \right\}. \quad (3.9)$$

This definition means that $\lambda_{m_s} + k_s$ (resp. $\lambda_{m_s} - \check{k}_s^{(\nu)}$) is the $(p_s + 1 - s)$ -th smallest (resp. the ν -th largest) integer not in the set $S(\lambda)$ which is bigger (resp. smaller) than λ_{m_s} . For convenience we set $\check{k}_i^{(0)} = 0$ and we simply denote $\check{k}_i = \check{k}_i^{(1)}$ for $i = 1, 2, \dots, r$.

A simple general way to compute k_i is the following procedure: First set $S = S(\lambda)$. Suppose we have computed $k_r, k_{r-1}, \dots, k_{i+1}$. To compute k_i , we count the integers in the set S starting with λ_{m_i} and stop at the first integer, say k , not in S . Then $k_i = k - \lambda_{m_i}$. **Now add k into the set S , and continue.**

The computation of $\check{k}_i^{(\nu)}$ is much simpler: count the integers downward in the set $S(\lambda)$ starting with λ_{m_i} until we find ν integers not in $S(\lambda)$. Say we stop at the integer \check{k} , then $\check{k}_i^{(\nu)} = \lambda_{m_i} - \check{k}$.

For example, if λ is as in (3.5), then

$$(k_1, k_2, k_3, k_4) = (10, 2, 2, 1) \text{ and } (\check{k}_1, \check{k}_2, \check{k}_3, \check{k}_4) = (1, 5, 1, 1). \quad (3.10)$$

If $\lambda \in P_r$ (not necessarily in D_r), we can still compute k_i and $\check{k}_i^{(\nu)}$ in the above way, but the difference lies in that the k_i 's are computed in the order that each time we compute k_i with λ_{m_i} being the largest among all those λ_{m_i} 's, the corresponding k_i 's of which are not yet computed.

Lemma 3.4. *Suppose $\lambda \in D_r$.*

(1) *For each $s \in \{1, 2, \dots, r\}$, k_s is the smallest positive integer such that*

$$(\lambda + \theta_t k_t d_{m_t, n_t}) + k_s d_{m_s, n_s} \text{ is regular for } \theta_t \in \{0, 1\}, s < t \leq r, \quad (3.11)$$

where $d_{i,j} \in \mathbb{Z}^{m+n}$ is the $(m+n)$ -tuple whose entries are zero except the i -th and j -th entries which are 1.

(2) *For each $s \in \{1, 2, \dots, r\}$, \check{k}_s is the smallest positive integer such that*

$$\lambda - \check{k}_s d_{m_s, n_s} \text{ is regular.} \quad (3.12)$$

(3) *The tuple $(k_r, k_{r-1}, \dots, k_1)$ is the lexicographically smallest tuple of positive integers such that for all $\theta = (\theta_1, \theta_2, \dots, \theta_r) \in \{0, 1\}^r$, $\lambda + \sum_{s=1}^r \theta_s k_s d_{m_s, n_s}$ is regular. Thus $(k_r, k_{r-1}, \dots, k_1)$ is the tuple satisfying [2, Main Theorem].*

Proof. See [21, Lemma 3.3]. It can also be proved directly using the definitions (3.8) and (3.9). \square

Following [2] (see also [25, 21]), we define the *raising operator* R_{m_s, n_s} and the *lowering operator* L_{m_s, n_s} on P_r by

$$R_{m_s, n_s}(\lambda) = \lambda + k_s d_{m_s, n_s}, \quad L_{m_s, n_s}(\lambda) = \lambda - \check{k}_s d_{m_s, n_s}, \quad (3.13)$$

for $s = 1, 2, \dots, r$ and $\lambda \in P_r$. It is straightforward to verify that the composition of ν copies of the operator L_{m_s, n_s} is

$$L_{m_s, n_s}^\nu(\lambda) = \lambda - \check{k}_s^{(\nu)} d_{m_s, n_s} \text{ for } \nu \geq 1. \quad (3.14)$$

Let $\theta = (\theta_1, \theta_2, \dots, \theta_r) \in \mathbb{N}^r$, where $\mathbb{N} = \{0, 1, \dots\}$. We define

$$R_\theta(\lambda) = (R_{m_r, n_r}^{\theta_r} \circ R_{m_{r-1}, n_{r-1}}^{\theta_{r-1}} \circ \dots \circ R_{m_1, n_1}^{\theta_1})(\lambda)^+, \quad (3.15)$$

$$L'_\theta(\lambda) = (L_{m_r, n_r}^{\theta_r} \circ L_{m_{r-1}, n_{r-1}}^{\theta_{r-1}} \circ \dots \circ L_{m_1, n_1}^{\theta_1})(\lambda)^+, \quad (3.16)$$

where in general λ^+ denotes the unique dominant element which is W -conjugate to λ .

Remark 3.5. *We remark that the definition of lowering operator defined here is different from that defined in [2, §3-f]. In [2, §3-f], \check{k}_s appearing in (3.13) needs to satisfy the following condition instead of condition (3.12):*

$$(\lambda - \theta_p \check{k}_p d_{m_p, n_p}) - \check{k}_s d_{m_s, n_s} \text{ is regular for all } 1 \leq p < s, \theta_p \in \{0, 1\}.$$

From definitions (3.8) and (3.15), by induction on $\#\{s \in [1, r] \mid \theta_s = 1\}$, we can prove (cf. [2, Main Theorem] or Lemma 3.4(3))

$$R_\theta(\lambda) = \left(\lambda + \sum_{s=1}^r \theta_s k_s d_{m_s, n_s}\right)^+ \text{ for } \theta \in \{0, 1\}^r. \quad (3.17)$$

Also from (3.16) and (3.14), by induction on $\#\{s \in [1, r] \mid \theta_s \neq 0\}$, we can prove

$$L'_\theta(\lambda) = \left(\lambda - \sum_{s=1}^r \check{\mathbf{k}}_s^{(\theta_s)} d_{m_s, n_s} \right)^+ \text{ for } \theta \in \mathbb{N}^r, \quad (3.18)$$

where $\check{\mathbf{k}}_s^{(\theta_s)}$ is define as follows: first set

$$\lambda^{(0)} = \lambda \text{ and } \lambda^{(s)} = L_{m_s, n_s}^{\theta_s} \circ \cdots \circ L_{m_1, n_1}^{\theta_1}(\lambda) \text{ if } s \in [1, r], \quad (3.19)$$

then $\check{\mathbf{k}}_s^{(\theta_s)}$ is $(\check{k}_s^{(\theta_s)})^{\lambda^{(s-1)}}$ (which is $\check{k}_s^{(\theta_s)}$ defined by $\lambda^{(s-1)}$, cf. Convention 3.1).

One can also observe that

$$\lambda = L'_\theta(\mu) \text{ or } \lambda = R_\theta(\mu) \implies \bar{t}^\lambda = \bar{t}^\mu \text{ (cf. (3.38)).} \quad (3.20)$$

3.3. Composition factors of Kac-modules. Let $\lambda, \mu \in \mathbb{Z}_+^{m|n}$. We define $a_{\lambda, \mu} = [V_\lambda, L_\mu]$ to be the multiplicity of irreducible module L_μ in the composition series of the Kac-module V_λ . If $a_{\lambda, \mu} \neq 0$, we say L_μ is a *composition factor* of V_λ , and μ is a *primitive weight* of V_λ .

The following theorem, which was conjectured in [25, Conjecture 4.1], is due to Brundan [2, Main Theorem].

Theorem 3.6. *Let $\mu \in \mathbb{Z}_+^{m|n}$ be an r -fold atypical dominant integral weight. Then $a_{\lambda, \mu} \leq 1$ for all λ and*

$$a_{\lambda, \mu} = 1 \iff \lambda = R_\theta(\mu) \text{ for some } \theta \in \{0, 1\}^r. \quad (3.21)$$

As stated in the introduction, this theorem is useful in understanding structures of Kac-modules, but it is still desirable to give a closed formula to compute the composition factors of the Kac-module V_λ , for a given λ . Below we shall implement this theorem to derive such a formula (see Theorem 3.13).

Denote

$$\Theta_r = \{\theta = (\theta_1, \theta_2, \dots, \theta_r) \in \mathbb{N}^r \mid \theta_s \leq s \text{ for } s = 1, 2, \dots, r\}, \quad (3.22)$$

a subset of \mathbb{N}^r of cardinality $(r+1)!$.

Definition 3.7. *Let $\lambda \in D_r$. We define Θ^λ to be the subset of Θ_r consisting of $\theta = (\theta_1, \theta_2, \dots, \theta_r)$ satisfying the following conditions.*

For $s = 1, 2, \dots, r$, if $\theta_s \neq 0$, then

$$c_{s-\theta_s, s} \neq \mathbf{c}, \text{ and } c_{s+1-\theta_s, s} \neq \mathbf{n}, \quad (3.23)$$

and for all $p \in [s+1-\theta_s, s-1]$,

$$\theta_p \leq \theta_s - s + p, \text{ and equality implies } c_{p, s} = \mathbf{c}, \quad (3.24)$$

and furthermore in case $c_{p, s} \neq \mathbf{n}$,

$$\theta_p \neq 0, \text{ or } \exists p' \in [p+1, s] \text{ such that } c_{p, p'} = \mathbf{q} \text{ and } \theta_{p'} \geq p' + 1 - p, \quad (3.25)$$

$$\text{with the exception that } p = s+1-\theta_s < s \text{ and } c_{p, s} = \mathbf{q} \implies \theta_p = 0. \quad (3.26)$$

Convention 3.8. *We use the convention that if an undefined notation appears in an expression, then this expression is omitted; for instance in (3.23) condition $c_{s-\theta_s, s} \neq \mathbf{c}$ is omitted in case $\theta_s = s$.*

Remark 3.9. (1) If $c_{s,t} = \mathbf{c}$ for all $s < t$ (in this case the atypical roots of λ are called totally \mathbf{c} -related), then the first condition of (3.23) implies $s = \theta_s$ and condition (3.25) implies $\theta_p \neq 0$. Thus

$$\Theta^\lambda = \{(1, 2, \dots, p, 0, \dots, 0) \mid p = 0, 1, \dots, r\} \text{ is of cardinality } r + 1. \quad (3.27)$$

(2) If $c_{s,t} = \mathbf{n}$ for all $s < t$ (in this case the atypical roots of λ are called totally \mathbf{n} -related), then the second condition of (3.23) implies $s + 1 - \theta_s = s$, i.e., $\theta_s = 1$, and other conditions are all missing since p does not exist in this case. So

$$\Theta^\lambda = \{0, 1\}^r \text{ is of cardinality } 2^r. \quad (3.28)$$

Let us describe how to determine all elements of Θ^λ in general. To do this, we need some more notations. For $1 \leq s \leq t \leq r$, denote by (roughly speaking, $\lambda^{(s,t)}$ defined below only keeps those entries of λ ranging from the s -th atypical entry to the t -th atypical entry)

$$\lambda^{(s,t)} \quad (3.29)$$

the $(t - s + 1)$ -fold atypical weight (for some Lie superalgebra $\mathfrak{gl}_{k|l}$ with $k \leq m$, $l \leq n$) obtained from λ by deleting all entries λ_i for $1 \leq i < m_t$ or $m_{s-1} < i \leq m$ (set $m_0 = m + 1$), and deleting $\lambda_{\bar{\eta}}$ for $m < \bar{\eta} < n_{s-1}$ (set $n_0 = m$) or $n_t < \bar{\eta} \leq \bar{n}$. For instance, if λ is as in (3.5), then

$$\lambda^{(2,3)} = (\underline{10}, \underline{7}, 6, 4 \parallel 5, \underline{7}, 8, \underline{10}), \quad \lambda^{(3,4)} = (\underline{15}, 11, \underline{10} \parallel 8, \underline{10}, \underline{15}), \quad (3.30)$$

where here and below, the underlined entries are the atypical entries, i.e., entries corresponding to the atypical roots. Then we can define $\Theta^{\lambda^{(s,t)}} \subset \Theta_{t-s+1}$ by Definition 3.7. Let $\theta \in \mathbb{N}^r$ and $S \subset [1, r]$. We denote

$$\theta_S \in \mathbb{N}^{\#S} \quad (3.31)$$

to be the element obtained from θ by deleting entries θ_i for $i \notin S$.

By Definition 3.7, we immediately obtain the following lemma which can be used to determine all elements of Θ^λ (by the procedure of induction on r).

Lemma 3.10. *Let*

$$M = \{s \in [1, r + 1] \mid c_{s-1,r} \neq \mathbf{c}, c_{s,r} \neq \mathbf{n}\}, \quad (3.32)$$

(note that Convention 3.8 means that $r + 1 \in M$), and for $s \in M$ denote

$$\Theta^{\lambda,s} = \begin{cases} \{\theta \in \Theta_r \mid \theta_{[2,r]} \in \Theta^{\lambda^{(2,r)}}, \theta_1 = 0, \theta_r = r\} & \text{if } s = 1, c_{1,r} = \mathbf{q}, \\ \{\theta \in \Theta_r \mid \theta_{[1,r-1]} \in \Theta^{\lambda^{(1,r-1)}}, \theta_1 \neq 0, \theta_r = r\} & \text{if } s = 1, c_{1,r} = \mathbf{c}, \\ \{\theta \in \Theta_r \mid \theta_{[1,s-1]} \in \Theta^{\lambda^{(1,s-1)}}, \\ \quad \theta_{[s,r]} \in \Theta^{\lambda^{(s,r)}}, \theta_r = r + 1 - s\} & \text{if } 2 \leq s \leq r, \\ \{\theta \in \Theta_r \mid \theta_{[1,r-1]} \in \Theta^{\lambda^{(1,r-1)}}, \theta_r = 0\} & \text{if } s = r + 1. \end{cases} \quad (3.33)$$

Then

$$\Theta^\lambda = \bigcup_{s \in M} \Theta^{\lambda,s} \quad (\text{disjoint union}). \quad (3.34)$$

Example 3.11. Suppose λ is as in (3.5). Then Θ^λ contains the following 14 elements:

$$\begin{aligned} & (0, 0, 0, 0), (1, 0, 0, 0), (1, 2, 0, 0), (0, 0, 1, 0), (1, 0, 1, 0), (1, 2, 1, 0), (1, 0, 3, 0), \\ & (0, 0, 0, 1), (1, 0, 0, 1), (1, 2, 0, 1), (0, 0, 1, 1), (1, 0, 1, 1), (1, 2, 1, 1), (1, 0, 3, 1). \end{aligned} \quad (3.35)$$

The following interesting fact was observed in [6].

Lemma 3.12. *If $c_{s,t} = \mathbf{q}$ for all $s < t$ (in this case the atypical roots of λ are called totally \mathbf{q} -related), then*

$$\#\Theta^\lambda = \frac{1}{r+2} \binom{2r+2}{r+1} = C_{r+1}, \quad (3.36)$$

where $C_r = \frac{1}{r+1} \binom{2r}{r}$ is the well-known r -th Catalan number.

Proof. It is easy to check (3.36) if $r = 1, 2$. Suppose $r \geq 3$. Note from (3.33) and condition (3.26) that if $2 \leq s \leq r$, then for $\theta \in \Theta^{\lambda,s}$, one has $\theta_s = 0$, i.e., θ is determined by $\theta_{[1,s-1]}$ and $\theta_{[s+1,r-1]}$, namely, θ has $C_s C_{r-s}$ choices by the inductive assumption. Thus

$$\#\Theta^\lambda = \sum_{s=1}^{r+1} \#\Theta^{\lambda,s} = C_{r-1} + \sum_{s=2}^r C_s C_{r-s} + C_r = C_{r+1},$$

where the last equality is a known combinatorial identity, whose proof is omitted. \square

The main result of this paper is the following theorem.

Theorem 3.13. *Let $\lambda \in \mathbb{Z}_+^{m|n}$ be an r -fold atypical dominant integral weight. Then*

$$\text{the set of primitive weights of } V_\lambda = \{L'_\theta(\lambda) \mid \theta \in \Theta^\lambda\}. \quad (3.37)$$

We shall prove Theorem 3.13 in Section 5. Let us look at the following example.

Example 3.14. *Suppose again λ is as in (3.5). It is hardly possible to use Theorem 3.6 to determine the primitive weights of V_λ . However this job can be easily done by Theorem 3.13. Using Example 3.11, (3.10) and (3.15), the primitive weights of V_λ are the following 14 weights: the first 7 weights are (cf. (3.20))*

$$\begin{aligned} &(\underline{15}, 11, \underline{10}, \underline{7}, 6, 4, \underline{3} \parallel \underline{3}, 5, \underline{7}, 8, \underline{10}, \underline{15}), (\underline{15}, 11, \underline{10}, \underline{7}, 6, 4, \underline{2} \parallel \underline{2}, 5, \underline{7}, 8, \underline{10}, \underline{15}), \\ &(\underline{15}, 11, \underline{10}, 6, 4, \underline{2}, \underline{1} \parallel \underline{1}, \underline{2}, 5, 8, \underline{10}, \underline{15}), (\underline{15}, 11, \underline{9}, \underline{7}, 6, 4, \underline{3} \parallel \underline{3}, 5, \underline{7}, 8, \underline{9}, \underline{15}), \\ &(\underline{15}, 11, \underline{9}, \underline{7}, 6, 4, \underline{2} \parallel \underline{2}, 5, \underline{7}, 8, \underline{9}, \underline{15}), (\underline{15}, 11, \underline{9}, 6, 4, \underline{2}, \underline{1} \parallel \underline{1}, \underline{2}, 5, 8, \underline{9}, \underline{15}), \\ &(\underline{15}, 11, \underline{7}, 6, 4, \underline{2}, \underline{1} \parallel \underline{1}, \underline{2}, 5, \underline{7}, 8, \underline{15}), \end{aligned} \quad (3.38)$$

and the other 7 weights are obtained from (3.38) by changing 15 to 14 in all positions.

Remark 3.15. *One may regard Theorem 3.13 as the “converse” of Theorem 3.6 in the sense that Theorem 3.6 computes $a_{\lambda,\mu}$ for a given μ while Theorem 3.13 computes $a_{\lambda,\mu}$ for a given λ .*

4. A CONJECTURE OF HUGHES ET AL

The purpose of this section is prove a conjecture of Hughes, King and van der Jeugt [6]. We shall briefly recall some notions, which will be used throughout the section. For more details, we refer to [6, 24] (see also [4, 23, 25, 19, 18]).

4.1. Composite Young diagram. Let $\lambda = (\lambda^1 \mid \lambda^2)$ be an r -fold atypical dominant integral weight, written in terms of notation (2.1), where $\lambda^1 = (\lambda'_1, \lambda'_2, \dots, \lambda'_m)$ and $\lambda^2 = (\lambda'_1, \lambda'_2, \dots, \lambda'_n)$. By the statements after (2.3), we can assume

$$\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_m \geq 0 \geq \lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_n, \quad (4.1)$$

Furthermore we can suppose λ'_m and $-\lambda'_1$ are large enough in order to be able to perform boundary strip removals (see Subsection 4.3). Since both λ^1 and $-(\lambda^2)^R := (-\lambda'_n, -\lambda'_{n-1}, \dots, \lambda'_1)$ are partitions, this allows us to associate a *composite Young diagram*

in [6], this rule was implicitly applied. We realize that if this rule is not included in the definition, one would produce two “codes” such as $(\begin{smallmatrix} 1 & 0 & 3 & 0 \\ 3 & & & \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 3 & 3 & 0 \\ 3 & & & \end{smallmatrix})$ (cf. (4.2)) which would correspond to the same sequence of boundary strip removals (see Subsection 4.3).

(2) From the proof of Theorem 3.13, we shall see that a code must satisfy the following rule stronger than rule (vii):

(vii)' If a column has two nonzero labels, then all labels, except possibly the top label, of this column must appear in the next column.

Remark 4.5. Rules (iv), (vi) and (vii)' imply the following rule stronger than rule (vi):

(vi)' Let $s < t < u < v$ such that columns s, t, u, v contain labels a, b, a, b respectively with $a \neq 0 \neq b$. If $a < b$ then columns s and u must contain b ; if $a > b$ then columns t and v must contain a .

4.3. Boundary strip removal. For a code μ_c for λ , it corresponds to a weight μ defined as follows.

First μ_c corresponds to a sequence of *coordinated boundary strip removals* [6] defined by: For each $s = 1, 2, \dots, r$, if label s appears in μ_c , first we set

$$a_s = \min\{p \in [1, r] \mid \text{label } s \text{ appears in column } p\}, \quad (4.3)$$

(i.e., column a_s is the first column which contains label s), and we use notation (2.6), then a *coordinated boundary strip removal starting from the s -th atypical γ_s and ending at the a_s -th atypical γ_{a_s}* is performed, that is, two boundary strip removals are simultaneously performed on F^{λ^1} and on $F^{-(\lambda^2)^R}$ which start respectively from the m_s -th row of F^{λ^1} and the $(\bar{n} + 1 - n_s)$ -th column of $F^{-(\lambda^2)^R}$ and continue until they pass the m_{a_s} -th row of F^{λ^1} and the $(\bar{n} + 1 - n_{a_s})$ -th column of $F^{-(\lambda^2)^R}$, then continue until the remaining composite Young diagram is standard. Then μ is defined to be the weight whose composite Young diagram is the remaining diagram.

Thus we obtain a correspondence

$$\mu_c \mapsto \mu. \quad (4.4)$$

Example 4.6. If λ is as in Example 4.1, and μ_c is the 7-th code $(\begin{smallmatrix} 1 & 3 & 3 & 0 \\ 3 & & & \end{smallmatrix})$ in (4.2), then the boxes in the sequence of coordinated boundary strip removals are labeled in the diagram $F^{(\lambda^1|\lambda^2)}$ of Example 4.1. Thus

$$\mu = (8, 5, 3, 2, 1, 0, 0 \mid 0, 0, -2, -3, -3, -9). \quad (4.5)$$

which is the same as the last weight in (3.38).

Similarly, an element $\theta \in \Theta^\lambda$ also corresponds to a sequence of coordinated boundary strip removals, such that for each $s = 1, 2, \dots, r$, if $\theta_s \neq 0$, then a coordinated boundary strip removal starting from γ_s and ending at $\gamma_{s+1-\theta_s}$ is performed. Then we see that the remaining diagram is the composite Young diagram of μ , where $\mu = L'_\theta(\lambda)$.

Example 4.7. If $\theta = (1, 0, 3, 0)$ as in (3.35), then again we have the sequence of coordinated boundary strip removals in Example 4.1, and μ is as in (4.5).

Remark 4.8. Parallel to boundary strip removals, one can introduce the notion of boundary strip additions (we omit the precise definition here since we shall not use this later), such that for all r -fold atypical $\mu \in \mathbb{Z}_+^{m|n}$, each $\theta' \in \{0, 1\}^r$ “corresponds” (under some regulations) to a sequence of coordinated boundary strip additions performed on

the composite Young diagram of μ . The resulting diagram is the diagram of λ , where $\lambda = R_{\theta'}(\mu)$ (cf. Theorem 3.6). For example, if μ is the weight corresponding to the un-labeled boxes in Example 4.1 (i.e., μ is the weight in (4.5)), and $\theta' = (1, 1, 0, 0)$, then $\lambda = R_{\theta'}(\mu)$ is the weight in the example, where labels 3 and 1 in the diagram shall be changed to 1 and 2 respectively in order that the labeled boxes correspond to the boundary strip additions (cf. Lemma 5.5 below).

4.4. A conjecture of Hughes et al. As an application of Theorem 3.13, we prove the following theorem which was a conjecture put forward by Hughes, King and van der Jeugt in [6] as the result of in depth research carried out by the authors over several years time.

Theorem 4.9. *The correspondence (4.4) is a 1 - 1 correspondence between the set of primitive weights of V_λ and the set of permissible codes for λ .*

Partial result of this theorem was obtained in [19] (also cf. [18]), where it was proved that an “unlinked code” (i.e., a code that does not have two columns with the same nonzero top label) corresponds to a “strongly” primitive weight (i.e., a primitive weight whose primitive vector is a highest weight vector in the Kac-module); moreover the primitive vector corresponding to the unlinked code is precisely constructed.

Proof of Theorem 4.9. Let

$$\mathcal{C} = \text{the set of permissible codes for } \lambda. \quad (4.6)$$

We establish a 1 - 1 correspondence between \mathcal{C} and Θ^λ (then Theorem 4.9 follows from Theorem 3.13). Let $\mu_c \in \mathcal{C}$ be a code. We define $\theta \in \Theta^\lambda$ as follows: for $s = 1, 2, \dots, r$,

$$\theta_s \neq 0 \iff \text{label } s \text{ appears in the code } \mu_c, \text{ and in this case} \quad (4.7)$$

$$\theta_s = s + 1 - a_s, \quad (4.8)$$

where a_s defined in (4.3) is the first number whose column contains label s (and obviously column s is the last number whose column contains label s as the top label: note that if a label s appears in a code μ_c , it must be the top label of column s). For example, if μ_c is a code in (4.2), then θ is a corresponding element in (3.35).

We want to prove that θ is indeed in Θ^λ by verifying each condition of (3.23)–(3.26). This will be done by several claims. So suppose $\theta_s \neq 0$ for some $s \in [1, r]$.

Claim 1. The second condition of (3.23) holds, namely, $c_{a_s, s} \neq \mathbf{n}$.

Since label s appears in column a_s , if it is the top label, by rule (i) and the fact that \mathbf{q} -relationship is transitive we have $c_{a_s, s} = \mathbf{q}$. Otherwise, let p be the top label of column a_s , then $p < s$ and $c_{p, s} = \mathbf{c}$ by rule (iii). But rule (i) says that $c_{a_s, p} = \mathbf{q}$, which together with the relation $c_{p, s} = \mathbf{c}$ implies that $c_{a_s, s} = \mathbf{c}$. This proves Claim 1.

Claim 2. The first condition of (3.23) holds, namely, $c_{a_s-1, s} \neq \mathbf{c}$.

Suppose conversely $c_{a_s-1, s} = \mathbf{c}$.

Case (a): First suppose $c_{a_s-1, a_s} = \mathbf{c}$. Since label s appears in column a_s of μ_c , by rules (ii) and (iv), all labels of column a_s must appear in column $a_s - 1$, in particular s appears in column $a_s - 1$. This in turn contradicts definition (4.3). Thus this case does not occur.

Case (b): Suppose $c_{a_s-1, a_s} \neq \mathbf{c}$. If $c_{a_s, s} = \mathbf{q}$, then this and the relation $c_{a_s-1, a_s} \neq \mathbf{c}$ imply that $a_{a_s-1, s} \neq \mathbf{c}$ by definition (3.4), a contradiction with the assumption. Thus $c_{a_s, s} = \mathbf{c}$ by Claim 1.

If $c_{p,s} = \mathbf{c}$ for all $p \in [a_s, s-1]$, then by rule (ii), label s must appear in column $a_p - 1$, a contradiction with the definition (4.3). So $c_{p',s} \neq \mathbf{c}$ for some $p' \in [a_s, s-1]$. Set

$$p' = \min\{p' \in [a_s, s-1] \mid c_{p',s} \neq \mathbf{c}\} \quad (4.9)$$

to be the smallest whose corresponding atypical root $\gamma_{p'}$ of λ is not \mathbf{c} -related to the s -th atypical root γ_s . Then $c_{p',s} = \mathbf{q}$ (otherwise $c_{p'-1,s}$ cannot be \mathbf{c} , contradicting (4.9)). From this and definition (4.9), we have

$$c_{p,p'} = c_{p,s} = \mathbf{c} \quad \text{for all } p \in [a_s - 1, p' - 1]. \quad (4.10)$$

So rules (ii) and (iv) show that

$$\text{all nonzero labels in column } p' \text{ must appear in columns } a_s \text{ and } a_s - 1. \quad (4.11)$$

Thus the following subclaim means that s appears in column $a_s - 1$, this contradiction with definition (4.3) implies Claim 2.

Subclaim 2a). The top label of column p' is s .

To prove this subclaim, suppose u is the last label of column a_s . First assume that $u \neq s$. Then $s < u$ and $c_{s,u} = \mathbf{c}$ by rule (iii). This, together with the relation $c_{p',s} = \mathbf{q}$ and (4.10), shows that $c_{p,u} = \mathbf{c}$ for $p \in [a_s - 1, p']$, which in turn implies that u appears in column p by rule (vii). Thus u appears in column p' and it is not the top label of p' by rule (i). So let u' be the top label of p' . Then $c_{p',u'} = \mathbf{q}$ and so $c_{s,u'} = \mathbf{q}$ (if $s \leq u'$) or $c_{u',s} = \mathbf{q}$ (if $u' < s$) by the fact that $c_{p',s} = \mathbf{q}$. By rule (iii), two labels whose corresponding atypical roots are \mathbf{q} -related cannot appear in the same column, but both s and u' appear in column a_s by (4.11). We have $u' = s$, and thus obtain the subclaim in this case.

Next assume $s = u$ is the last label of column a_s . Then rule (vii) implies that s appears in column p for all $p \in [a_s - 1, p']$. In particular, s must be the top label of column p' . The subclaim is proved.

(The proof of the subclaim also shows that a code satisfies rule (vii)' in Remark 4.4.)

Claim 3. Condition (3.24) holds.

First suppose $\theta_p > \theta_s - s + p$ for some $p \in [a_s, s-1]$. Then we have $a_p < a_s < p < s$ and columns a_p, a_s, p, s contain labels p, s, p, s respectively. By rule (vi)' in Remark 4.5, s appears in column a_p (which is $< a_s$), a contradiction with definition (4.3). Next suppose $\theta_p = \theta_s - s + p$. Then $a_p = a_s$ and so both p and s appear in column a_s . By rule (iii), we must have $c_{p,s} = \mathbf{c}$. This proves Claim 3.

Claim 4. Condition (3.25) holds.

Assume conversely for some $p \in [a_s, s-1]$, $c_{p,s} \neq \mathbf{n}$, but

$$\begin{aligned} \theta_p &= 0 \text{ and no } p' \in [p+1, s] \text{ with } c_{p,p'} = \mathbf{q} \text{ and } \theta_{p'} \geq p' + 1 - p, \\ \text{and } p &\neq a_s \text{ or } c_{a_s,s} \neq \mathbf{q} \quad (\text{cf. condition (3.26)}). \end{aligned} \quad (4.12)$$

Condition (4.12) implies

$$\text{column } p \text{ of } \mu_c = 0. \quad (4.13)$$

Claim 1 allows us to consider the following two cases:

Case (a): First suppose $c_{a_s,s} = \mathbf{q}$. Then rule (v) and (4.13) mean that $c_{p,s} \neq \mathbf{q}$, i.e., $c_{p,s} = \mathbf{c}$. This together with rule (ii) and (4.13) implies that (cf. the arguments in Case (b) of Claim 2) there exists $p_1 \in [p+1, s-1]$ such that $c_{p_1,s} \neq \mathbf{c}$. Let

$$p_1 = \min\{p_1 \in [p+1, s-1] \mid c_{p_1,s} \neq \mathbf{c}\} \quad (4.14)$$

be the smallest whose corresponding atypical root γ_{p_1} of λ is not \mathbf{c} -related to γ_s . If $c_{p_1,s} = \mathbf{n}$, then from definition (3.4), it follows that $c_{p_1-1,s} \neq \mathbf{c}$, contradicting (4.14). Thus $c_{p_1,s} = \mathbf{q}$. Using this and definition (4.14), we have $c_{p',p_1} = c_{p',s} = \mathbf{c}$ for all $p' \in [p, p_1 - 1]$. This, together with the fact that column p_1 is nonzero (which is derived by rule (v) and the relation $c_{p_1,s} = \mathbf{q}$), contradicts (4.13) by rule (ii).

Case (b): Suppose $c_{a_s,s} = \mathbf{c}$. This time instead of defining p_1 in (4.14), we define p'_1 by

$$p'_1 = \min\{p'_1 \in [a_s, s-1] \mid c_{p'_1,s} \neq \mathbf{c}\}. \quad (4.15)$$

The fact (4.13) ensures that such p'_1 exists by using rule (ii) since column s is nonzero, and $c_{p'_1,s} = \mathbf{q}$ as in Case (a). Now rules (ii), (vii) and rule (vi)' in Remark 4.5 show that s appears in column p' for all $p' \in [a_s, p'_1]$. In particular $p'_1 < p$ by (4.13). Now by the same arguments in Case (a) (with a_s replaced with p'_1), one obtains a contradiction. This proves Claim 4.

Claim 5. Condition (3.26) holds.

To prove this, suppose $a_s < s$ and $c_{a_s,s} = \mathbf{q}$. Since s appears in column a_p (cf. the first statement after Claim 1), by rules (i) and (iii), s must be the top label of column a_s . This in particular implies that label a_s cannot appear anywhere in μ_c (cf. the first statement after (4.8)), i.e., $\theta_p = 0$ for $p = a_s$. Claim 5 is proved.

Denote the obtained θ by θ_{μ_c} . The above claims show that we have a map

$$\mathcal{C} \rightarrow \Theta^\lambda : \mu_c \mapsto \theta_{\mu_c}. \quad (4.16)$$

Conversely, suppose $\theta \in \Theta^\lambda$. We define a code $\mu_c \in \mathcal{C}$ as follows.

For $s = 1, 2, \dots, r$, if $\theta_s \neq 0$, this time we first set a_s to be $s + 1 - \theta_s$ (cf. (4.8)), then the code μ_c should satisfy: label s must appear in column s (as the top label) of μ_c , and appear in column a_s (columns a_s and s are respectively the first and last columns where label s appears); and for all $p \in [a_s + 1, s - 1]$, label s appears in column p if and only if $c_{p,s} = \mathbf{c}$, or $c_{p,s} = \mathbf{q}$ but there does not exist a smaller label $s' \in [p, s - 1]$ which appears in this column.

It is a little tedious but straightforward routine to check that the above uniquely defines a code μ_c , denoted by μ_c^θ , satisfying rules (i)–(vii), and that the correspondence

$$\theta \mapsto \mu_c^\theta \quad (4.17)$$

obtained in this way is the inverse of the map (4.16). We omit the details since they are mainly the reverse of the above arguments. \square

5. PROOF OF THE MAIN THEOREM

The aim of this section is to give a proof of Theorem 3.13. This will be done by several lemmas. Having given a proof of Theorem 4.9 helps us in understanding the arguments below.

Parallel to definition (3.9), we introduce r -tuples $(\underline{k}_1^{(\nu)}, \underline{k}_2^{(\nu)}, \dots, \underline{k}_r^{(\nu)})$, $\nu \geq 1$, of positive integers defined by

$$\underline{k}_s^{(\nu)} = \min \left\{ k > 0 \mid \# \left([\lambda_{m_s}, \lambda_{m_s} + k] \setminus S(\lambda) \right) = \nu \right\}. \quad (5.1)$$

Thus $\lambda_{m_s} + \underline{k}_s^{(\nu)}$ is the ν -th smallest integer bigger than λ_{m_s} and not in $S(\lambda)$, and the definition of k_s in (3.8) implies

$$k_s = \underline{k}_s^{(p_s+1-s)} \text{ for } s = 1, 2, \dots, r. \quad (5.2)$$

By Theorem 3.6, the proof of (3.37) is equivalent to proving that for r -fold atypical weights $\lambda, \mu \in \mathbb{Z}_+^{m|n}$,

$$\mu = L'_\theta(\lambda) \text{ for some } \theta \in \Theta^\lambda \iff \lambda = R_{\theta'}(\mu) \text{ for some } \theta' \in \{0, 1\}^r. \quad (5.3)$$

Convention 5.1. *For convenience, unless it is specified, we shall always use the same notation with a tilde to denote any element or concept associated with μ ; for instance, $\tilde{k}_i = k_i^\mu$ (recall (3.8) and Convention 3.1), $\tilde{p}_s = p_s^\mu$, $\tilde{m}_s = m_s^\mu$, etc.*

First suppose $\mu = L'_\theta(\lambda)$ for some $\theta \in \Theta^\lambda$. For $s = 1, 2, \dots, r$, we define

$$a_s = s + 1 - \theta_s, \quad (5.4)$$

$$N_{s,p} = \#\{p' \in [a_s, p-1] \mid \theta_{p'} \neq 0, a_{p'} = a_s\} \text{ if } \theta_s \neq 0, \quad (5.5)$$

for $p \in [a_s, s-1]$, and set $N_{s,p} = -1$ if $\theta_s = 0$. We simply denote $N_s = N_{s,s}$. In terms of the code μ_c^θ defined in (4.17), a_s defined here is the first number whose column contains label s in the code μ_c^θ (cf. (4.3)), and $N_{s,p}$ is the number of those labels p' which is smaller than p and which first appear in column a_s (“first appearance” means “not appear in a smaller column”).

Lemma 5.2. *If $\theta_s \neq 0$, then (recall definition $\ell_{s,t}$ in (3.4))*

$$a_s \leq a_p \leq p \text{ for all } p \in [a_s, s-1] \text{ with } \theta_p \neq 0, \quad (5.6)$$

$$N_s = s - a_s - \ell_{a_s,s} \text{ and} \quad (5.7)$$

$$N_{s,p} < N_s \text{ if } p \in [a_s, s-1]. \quad (5.8)$$

Proof. The first equation follows from condition (3.24) and the last from definition (5.5). The second equation can be proved by induction on s as follows.

Set $N'_s = s - a_s - \ell_{a_s,s}$. If $N'_s = 0$, i.e., $c_{a_s,s} = \mathbf{q}$, then (5.7) follows from condition (3.26). Suppose $N'_s > 0$, i.e., $c_{a_s,s} = \mathbf{c}$. Condition (3.25) shows that there exists $p' \in [a_s, s-1]$ with $\theta_{p'} \neq 0$ such that $a_{p'} = a_s$. Let p' be the maximal such number. This definition of p' implies that $N_{s,p'} = N_s - 1$. But in fact we have $N_{s,p'} = N_{p'}$ from the definition (5.5). Thus $N_{p'} = N_s - 1$. Since $p' < s$, by inductive assumption on p' , we can suppose $N'_{p'} = N_{p'}$, i.e., $p' - a_s - \ell_{a_s,p'} = N_s - 1$. Thus the proof of (5.7) is reduced to proving

$$p' - \ell_{a_s,p'} = s - \ell_{a_s,s} - 1. \quad (5.9)$$

By condition (3.24),

$$c_{p',s} = \mathbf{c}, \text{ i.e., } \ell_{p',s} < s - p'. \quad (5.10)$$

By the maximal choice of p' , we must have $\ell_{p',s} = s - p' - 1$ (the arguments to prove this are similar to those given in Case (b) of Claim 2 in the proof of Theorem 4.9, thus omitted), which is equivalent to (5.9). \square

We define

$$\pi_s = \#\{p \in [s, r] \mid s = a_p = p + 1 - \theta_p\}, \quad (5.11)$$

$$\theta'_s = \begin{cases} 1 & \text{if } \exists p \in [1, s] \text{ such that } p \leq s < p + \pi_p, \\ 0 & \text{otherwise.} \end{cases} \quad (5.12)$$

Here π_s is the number of those labels p which first appear in column s of the code μ_c^θ (thus in particular $N_s = \pi_{a_s}$). Definition (5.12) means that $\theta'_s \neq 0$ if and only if column s is bigger than but “close” to a (unique) column p (where some labels first appear), here “close” means that the distinct $s - p$ is smaller than the number π_p of those labels first appearing in column p .

For example, if $\theta = (1, 0, 3, 0)$ in (3.35), then $\mu_c^\theta = (\begin{smallmatrix} 1 & 3 & 3 & 0 \end{smallmatrix})$ and

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (2, 0, 0, 0), \text{ and so } \theta' = (1, 1, 0, 0). \quad (5.13)$$

Note from definitions (5.4) and (5.12) that

$$\sum_{s \in [1, r]} \pi_s = \#\{s \in [1, r] \mid \theta_s \neq 0\} = \#\{s \in [1, r] \mid \theta'_s \neq 0\} =: N, \quad (5.14)$$

which is the number of nonzero θ_s 's (or nonzero θ'_s 's). Lemma 5.2 allows us to divide the index set $[1, r]$ into some subsets: $[1, r] = \mathcal{V}_1 \cup \mathcal{V}_2$ and

$$\mathcal{V}_2 = [\ell'_1, \ell_1] \cup [\ell'_2, \ell_2] \cup \dots \cup [\ell'_\nu, \ell_\nu] \quad (5.15)$$

for some ν and some $\ell'_1 \leq \ell_1 < \ell'_2 \leq \ell_2 < \dots < \ell'_\nu \leq \ell_\nu$, such that $\theta_p = 0$ for $p \in \mathcal{V}_1$ and

$$\theta_{\ell_i} \neq 0 \text{ and } \ell_i = \max\{p \in [\ell'_i, r] \mid a_p = \ell'_i\}, i = 1, \dots, \nu. \quad (5.16)$$

(The above means that ℓ_i is the largest label which first appears in column ℓ'_i of the code μ_c^θ , and this in turn means that all labels of column ℓ'_i first appear in this column.)

Lemma 5.3. $\lambda = R_{\theta'}(\mu)$.

Proof. Denote $\varphi = R_{\theta'}(\mu)$. By (3.20), it suffices to prove $T(\lambda) = T(\varphi)$ (cf. definition (3.3)), which in turn is equivalent to

$$T(\varphi) \subset T(\lambda) \quad (5.17)$$

by the dominance of φ . Equations (3.17) and (3.18) imply (cf. Convention 5.1)

$$T(\varphi) = \{\mu_{\tilde{m}_s} + \theta'_s \tilde{k}_s \mid s \in [1, r]\} \text{ and } T(\mu) = \{\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)} \mid s \in [1, r]\}, \quad (5.18)$$

where $\check{\mathbf{k}}_s^{(\theta_s)}$ is defined after (3.19). We want to prove (5.17). Nothing needs to be done if $N = 0$. So suppose $N > 0$. We first need to list the order of integers in $T(\mu)$ in order to determine $\mu_{\tilde{m}_s}$ for $s \in [1, r]$. This is done by the following claims.

Claim 1. Let $p \in [\ell'_i, \ell_i]$, $s \in [\ell'_j, \ell_j]$ for some $i < j$. Then $\lambda_{m_p} < \lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)}$.

If $\theta_s = 0$ there is nothing to do since $\lambda_{m_p} < \lambda_{m_s}$. Suppose $\theta_s > 0$. Then $p \leq \ell_i < \ell'_j \leq a_s$ and $c_{a_s-1, s} \neq \mathbf{c}$, i.e., $\ell_{a_s-1, s} \geq s - (a_s - 1) = \theta_s$. This and the definition of $\check{\mathbf{k}}_s^{(\theta_s)}$ show that $\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)}$ which is the θ_s -th largest integer smaller than $\lambda_{m_s} = \lambda_{m_s}^{(s-1)}$ and not in $S(\lambda^{(s-1)})$ (recall the definition of $\lambda^{(s)}$ in (3.19)), is bigger than λ_{m_p} . This is the claim.

Claim 2. Let $p \in [\ell'_i, \ell_i - 1]$ and $s = \ell_i$. Then $\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)} < \lambda_{m_p} - \check{\mathbf{k}}_p^{(\theta_p)}$.

Suppose $a_p = a_s$. The proof of Lemma 5.2 shows that $\ell_{p, s} < s - p$ (cf. (5.10)). Thus $\theta_s - \theta_p = s - p > \ell_{p, s}$, which in turn says that “the θ_s -th largest integer smaller than $\lambda_{m_s}^{(s-1)}$ and not in $S(\lambda^{(s-1)})$ ” is smaller than “the θ_p -th largest integer smaller than $\lambda_{m_p}^{(p-1)}$ and not in $S(\lambda^{(p-1)})$ ”. This is Claim 2 by the definition of $\check{\mathbf{k}}_s^{(\nu)}$.

If $a_s < a_p$, as in the proof of Claim 1 one can show that $\lambda_{m_{a_s}} < \lambda_{m_p} - \check{\mathbf{k}}_p^{(\theta_p)}$. But obviously $\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)} < \lambda_{m_{a_s}}$. Claim 2 is proved.

Claim 3. For $p, s \in [\ell'_i, \ell_i]$, $\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)} < \lambda_{m_p} - \check{\mathbf{k}}_p^{(\theta_p)}$ if and only if $a_s < a_p$ or $a_s = a_p$ but $p < s$.

This claim follows from the same arguments in the proofs of the above two claims.

The above three claims completely determine the order of the elements in the set $T(\mu)$. From this and the definition (5.12), one can see that $\theta'_p = 0$ if and only if $\theta_s = 0$

for some s such that $\mu_{\tilde{m}_p} = \lambda_{m_s}$. Thus to prove (5.17), by (5.18), we only need to consider elements $\mu_{\tilde{m}_s} + \theta'_s \tilde{k}_s$ of $T(\eta)$ with $\theta'_s \neq 0$. We denote

$$T_i(\mu) = \{\lambda_{m_s} - \check{\mathbf{k}}_s^{(\theta_s)} \mid s \in [\ell'_i, \ell_i]\}. \quad (5.19)$$

By using induction on ν , we shall only need to consider the set $T_i(\mu)$ for a particular i . By restricting to $[\ell'_i, \ell_i]$, we may regard $[\ell'_i, \ell_i]$ as the whole set $[1, r]$, i.e., $\ell'_i = 1$, $\ell_i = r$, in order to simplify notations. In such case, $\theta_r = r$, and $\theta'_1 = 1$. Then Claims 2 and 3 show that $\lambda_{m_r} - \check{\mathbf{k}}_r^{(r)}$ (in this case $\check{\mathbf{k}}_r^{(r)}$ is in fact $\tilde{k}_r^{(r)}$) is the smallest number in $T_i(\mu)$, thus it is the entry corresponding to the first atypical root of μ , i.e.,

$$\mu_{\tilde{m}_1} = \lambda_{m_r} - \check{\mathbf{k}}_r^{(r)}. \quad (5.20)$$

We denote

$$\eta = (L_{m_r, n_r}^r(\lambda))^+ = (\lambda - \check{\mathbf{k}}_r^{(r)} d_{m_r, n_r})^+ \quad (\text{cf. (3.14)}). \quad (5.21)$$

Then obviously we have

Claim 4. The first atypical entry of η is $\mu_{\tilde{m}_1} = \lambda_{m_r} - \check{\mathbf{k}}_r^{(r)}$, and the i -th atypical entry of η is $\lambda_{m_{i-1}}$ if $i > 1$.

Claim 5. $\tilde{k}_1 = \check{\mathbf{k}}_r^{(r)}$ (this means that the element $\mu_{\tilde{m}_1} + \theta'_1 \tilde{k}_1 = \lambda_{m_r}$ is indeed in the set $T(\lambda)$).

We prove this by induction on π_1 (cf. (5.11)). If $\pi_1 = 1$, i.e., there does not exist $s < r$ with $a_s = 1$. Then conditions (3.25) and (3.26) imply $c_{1,r} = \mathbf{q}$, i.e., $\ell_{1,r} = r - 1$, and $c_{s,r} = \mathbf{n}$, i.e., $\ell_{s,r} > r - s$, for all $s > 1$. This together with (5.21) and (5.20) in turn implies that $\ell_{1,s}^\eta < s - 1$ for all $s \in [2, r]$ (recall Convention 3.1). Note from Claim 4 and (5.18) that (recall Convention 5.1)

$$\tilde{\ell}_{1,s} \leq \max\{\ell_{1,p}^\eta \mid p \in [1, s]\} \leq s - 1. \quad (5.22)$$

This shows that $\tilde{c}_{1,s} = \mathbf{c}$ for all $s > 1$, i.e., $\tilde{p}_1 = r$. By (5.2), $\tilde{k}_1 = \tilde{\underline{k}}_1^{(r)}$, which is indeed equal to $\check{\mathbf{k}}_r^{(r)}$ from the definitions of $\check{\mathbf{k}}_r^{(r)}$ and $\tilde{\underline{k}}_1^{(r)}$ in (5.1).

If $\pi_1 > 1$. We choose $p < r$ to be the largest such that $a_p = 1$, then $\mu_{\tilde{m}_2} = \lambda_{m_p} - \check{\mathbf{k}}_p^{(\theta_p)}$ (i.e., which is the second smallest atypical entry of μ) by the maximal choice of p , and one can prove that $\tilde{c}_{1,2} = \mathbf{c}$ (i.e., $\tilde{\ell}_{1,2} = 0$). Using another induction on π_1 , one can again show that $\tilde{c}_{1,s} = \mathbf{c}$ for all $s > 1$. Thus as above we have $\tilde{k}_1 = \check{\mathbf{k}}_r^{(r)}$.

Claim 6. If $p \leq s < p + \pi_p$ for some p (such p must be unique), then $\tilde{k}_s = \check{\mathbf{k}}_t^{(\theta_t)}$, where t is the $(s + 1 - p)$ -th largest number such that $a_t = p$.

This claim follows from the proof of claim 5 by forgetting all labels not in the set $[p, t]$ (i.e., by restricting to $[p, t]$) and regarding $[p, t]$ as the whole set $[1, r]$.

Now (5.17) follows from Claim 6, and thus the lemma. \square

Remark 5.4. The proof of Lemma 5.3 can also be done by induction on $|\theta| := N = \#\{s \in [1, r] \mid \theta_s \neq 0\}$ in the following way: The definition of η in (5.21) shows that $\mu = L_{\theta^\eta}^\eta(\eta)$ for $\theta^\eta \in \Theta^\lambda$ with $\theta_i^\eta = \theta_i$ if $i < r$ and $\theta_r^\eta = 0$. So by the inductive assumption, $\eta = R_{\theta''}(\mu)$ for some $\theta'' \in \{0, 1\}^r$ such that $\theta''_1 = 0$. Then one can obtain that $\lambda = R_{m_1, n_1}(\eta) = R_{\theta'}(\mu)$, where $\theta'_1 = 1$ and $\theta'_i = \theta''_i$ if $i > 1$.

The arguments to be given below can be regarded as the “reverse” arguments of the above.

Next suppose $\lambda = R_{\theta'}(\mu)$ for some $\theta' \in \{0, 1\}^r$. First we note from the definition of (recall Convention 5.1) \tilde{p}_s in (3.6) that

$$s \leq t \leq \tilde{p}_t \leq \tilde{p}_s \quad \text{for all } t \in [s, \tilde{p}_s]. \quad (5.23)$$

As in (5.15), this allows us to divide $[1, r] = \mathcal{V}_1 \cup \mathcal{V}_2$, such that $\theta'_s = 0$ if $s \in \mathcal{V}_1$, and

$$\theta'_{\ell'_i} = 1 \text{ and } \tilde{p}_{\ell'_i} = \ell_i, \quad i = 1, 2, \dots, \nu. \quad (5.24)$$

(The above means that ℓ_i is the largest number such that all atypical roots $\tilde{\gamma}_s$, $\ell'_i < s \leq \ell_i$, of μ are \mathbf{c} -related to $\tilde{\gamma}_{\ell'_i}$.)

As above we can suppose $\nu = 1$ and $\ell'_i = 1$, $\ell_i = r$, i.e., $\theta'_1 = 1$, $p_1 = r$. This implies $\tilde{c}_{1,s} = \mathbf{c}$, i.e.,

$$\tilde{\ell}_{1,s} < s - 1 \text{ for all } s = 2, 3, \dots, r. \quad (5.25)$$

We set

$$\eta = R_{\tilde{m}_1, \tilde{n}_1}(\mu) = (\mu + \tilde{k}_1 d_{\tilde{m}_1, \tilde{n}_1})^+. \quad (5.26)$$

Lemma 5.5. *The i -th atypical entry of η is the $(i+1)$ -th atypical entry $\mu_{\tilde{m}_{i+1}}$ if $i < r$, or $\mu_{\tilde{m}_1, \tilde{n}_1} + \tilde{k}_1$ if $i = r$ (cf. Remarks 4.8, 5.4).*

Proof. This follows from (5.26) and definition (3.8). \square

Set $\theta'' \in \{0, 1\}^r$ with $\theta''_i = \theta'_{i+1}$ if $i < r$ and $\theta''_r = 0$. This and definition (3.15) imply

$$\lambda = R_{\theta'}(\mu) = R_{\theta''}(\eta). \quad (5.27)$$

By Lemma 5.5 and (5.26), we obtain that for all $i < r$, $p_i^\eta < r$ (recall Convention 3.1). Since $|\theta''| := \sum_{s=1}^r \theta''_s < |\theta'|$, we can use induction on $|\theta'|$ to suppose that there exists some $\theta^\eta \in \Theta^\lambda$ with $\theta_r^\eta = 0$, such that

$$\eta = L'_{\theta^\eta}(\lambda). \quad (5.28)$$

Note from definition (5.25) that for all $s < r$ we must have $\ell_{s,r}^\eta \leq r - 1$, which together with (5.28) in particular shows that

$$\ell_{1,r} = \ell_{1,r}^\lambda \leq \max\{\ell_{s,r}^\eta \mid s < r\} \leq r - 1. \quad (5.29)$$

Also (5.26) (cf. (5.28)) implies that $\mu = L_{m_r, n_r}^r(\eta)$ (a lowering operator $L_{i,j}$ is the “inverse operator” of some raising operator $R_{i',j'}$ in some sense), which together with (5.28) and the definition of L_θ in (3.16) implies

$$\mu = L_\theta(\lambda), \quad (5.30)$$

where $\theta \in \mathbb{N}^r$ is defined by $\theta_i = \theta_i^\eta$ if $i < r$ and $\theta_r = r$.

Lemma 5.6. $\theta \in \Theta^\lambda$.

Proof. The fact that $\theta^\eta \in \Theta^\lambda$ implies that conditions (3.23)–(3.26) hold for all $s < r$. That these conditions also hold for $s = r$ follows from the fact that $\theta_r = r$ and $c_{1,r} \neq \mathbf{n}$ (cf. (5.29)). \square

Proof of Theorem 3.13. Finally we return to the proof of Theorem 3.13. By Lemma 5.3, (5.30) and Lemma 5.6, we have (5.3). This implies the theorem. \square

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